

Additive Stabilizers for Unstable Graphs

Karthekeyan Chandrasekaran¹, Corinna Gottschalk², Jochen Könemann³, Britta Peis², Daniel Schmand², and Andreas Wierz²

¹University of Illinois Urbana-Champaign, Department of Industrial and Enterprise Systems Engineering,
karthe@illinois.edu

²RWTH Aachen University, School of Business and Economics,
{gottschalk,peis,schmand,wierz}@oms.rwth-aachen.de

³University of Waterloo, Department of Combinatorics & Optimization, jochen@uwaterloo.ca

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Abstract

Stabilization of graphs has received substantial attention in recent years due to its connection to game theory. Stable graphs are exactly the graphs inducing a matching game with non-empty core. They are also the graphs that induce a network bargaining game with a balanced solution. A graph with weighted edges is called stable if the maximum weight of an integral matching equals the cost of a minimum fractional weighted vertex cover. If a graph is not stable, it can be stabilized in different ways. Recent papers have considered the deletion or addition of edges and vertices in order to stabilize a graph. In this work, we focus on a fine-grained stabilization strategy, namely stabilization of graphs by fractionally increasing edge weights.

We show the following results for stabilization by minimum weight increase in edge weights (min additive stabilizer): (i) Any approximation algorithm for min additive stabilizer that achieves a factor of $O(|V|^{1/24-\epsilon})$ for $\epsilon > 0$ would lead to improvements in the approximability of densest- k -subgraph. (ii) Min additive stabilizer has no $o(\log |V|)$ approximation unless NP=P. Results (i) and (ii) together provide the first super-constant hardness results for any graph stabilization problem. On the algorithmic side, we present (iii) an algorithm to solve min additive stabilizer in factor-critical graphs exactly in poly-time, (iv) an algorithm to solve min additive stabilizer in arbitrary-graphs exactly in time exponential in the size of the Tutte set, and (v) a poly-time algorithm with approximation factor at most $\sqrt{|V|}$ for a super-class of the instances generated in our hardness proofs.

1 Introduction

Over the last two decades, algorithmic game theory has established itself as a vibrant and rich subarea of theoretical computer science as is evidenced by

several recent books (e.g., see [10, 28, 31]). A crucial driver in this development is the increasingly networked structure of today's society, and the impact this development has on the day-to-day interactions that humans engage in. Finding and analyzing graph-theoretic models for such networks is at the heart of the field of network exchange theory, and is captured by the two recent books [14, 20].

Social networks, and the interaction of individuals in those also motivated our work. Specifically, our interest started with [22], where Kleinberg & Tardos introduce *network bargaining* as a natural extension of Nash's classical two-player bargaining game [27] to the network setting. The players in Kleinberg and Tardos' game correspond to the vertices in an underlying graph $G = (V, E)$. Each $\{u, v\} \in E$ corresponds to a potential deal of given value $w_{uv} \geq 0$. Each player is allowed to interact with the neighbors to agree upon a sharing of the value on the edge between them and eventually arrive at a deal with at most one of her neighbours. Therefore, outcomes in network bargaining correspond to *matchings* $M \subseteq E$, and an *allocation* $y \in \mathbb{R}_+^V$ of $w(M)$ to the players. In particular, we want $y_u + y_v = w_{uv}$ for all $\{u, v\} \in M$, and $y_u = 0$ if u is not incident to an edge of M (u is *exposed*).

Kleinberg and Tardos introduce the concept of *stability*, and call an allocation y to be *stable* if $y_u + y_v \geq w_{uv}$ for *all* edges $\{u, v\} \in E$. Naturally extending Nash's bargaining solution, the authors define the *outside option* α_u of a player u given an allocation y as the largest value that u can *extract* from one of its neighbours. An allocation y is then deemed to be *balanced* if the value of each matching edge $\{u, v\} \in M$ is split according to Nash's bargaining condition: each player $a \in \{u, v\}$ receives its outside option α_a , and the remaining value of $\{u, v\}$ is divided equally among the players. One of Kleinberg and Tardos' main results is that balanced outcomes exist in a given network bargaining instance if and only if stable ones exist, and these can be computed efficiently.

Network bargaining is closely related to the cooperative *matching game* introduced by Shapley and Shubik [30], where the player set once more corresponds to the vertices of an underlying graph $G = (V, E)$, and the characteristic function assigns the maximum weight of a matching in $G[S]$ to each set $S \subseteq V$ of vertices. The *core* of an instance of this game consists of allocations $y \in \mathbb{R}_+^V$ of the weight $\nu(G, w)$ of a maximum-weight matching to the players such that $y_u + y_v \geq w_{uv}$ for all $\{u, v\} \in E$. Hence, core allocations exactly correspond to stable allocations in network bargaining (this observation was recently also made by Bateni et al. [5]).

A given instance of network bargaining therefore has a stable (and also a balanced) outcome if and only if the core of the corresponding matching game is non-empty. We state the classical maximum weight matching LP that has a variable x_e for each edge $e \in E$ (we use $x(\delta(v))$ as a convenient short-hand for $\sum_{e \in \delta(v)} x_e$, where $\delta(v)$ denotes the set of edges incident to v):

$$\nu_f(G, w) := \max \left\{ \sum_{e \in E} w_e x_e : x(\delta(v)) \leq 1 \text{ for all } v \in V, x \geq 0 \right\}. \quad (\text{P})$$

The linear programming dual of (P) has a variable y_v for each vertex $v \in V$, and a covering constraint for each edge $e \in E$:

$$\tau_f(G, w) := \min \left\{ \sum_{v \in V} y_v : y_u + y_v \geq w_{uv} \text{ for all } \{u, v\} \in E, y \geq 0 \right\}. \quad (\text{D})$$

Feasible solutions of (P) and (D) will henceforth be referred to as *fractional matchings* and *fractional w -vertex covers*, respectively. In the unit-weight special case, where $w = \mathbb{1}$, we will omit the argument w from the ν and τ notation for brevity. An immediate observation is that a given instance of network bargaining has a stable outcome iff the core of the corresponding matching game is non-empty iff $\nu(G, w) = \nu_f(G, w) = \tau_f(G, w)$, where the second equality follows from linear programming duality. In other words, stable outcomes exist iff LP (P) admits integral optimum solutions. We will call a (possibly weighted) graph *stable* if the induced network bargaining instance admits a stable outcome.

In [9], Bock et al. proposed the following *meta* problem: given an *unstable* graph G , modify G in the least *intrusive* way in order to attain stability. The authors focused on the concrete question of removing the smallest number of edges from G so that the resulting graph is stable. Bock et al. showed that this problem is as hard to approximate as the vertex cover problem, even if the underlying graph is factor critical (i.e., even if deleting any vertex from G yields a graph with a perfect matching). The authors complemented this negative result by presenting an approximation algorithm whose performance guarantee is proportional to the *sparsity* of the underlying graph.

Concurrently, Ahmadian et al. [1] and Ito et al. [19] proposed a *vertex-stabilizer* problem: given a graph $G = (V, E)$, find a minimum-cardinality set of vertices $S \subseteq V$ such that $G[V \setminus S]$ is stable. Both papers presented a combinatorial polynomial-time exact algorithm for this problem, and showed that the min cost variants of vertex stabilization are NP-hard. Ito et al. [19] proposed stabilizing a graph by adding a minimum number of vertices or edges. They showed that both of these problems are polynomial-time solvable. However, the minimum cost variant of stabilization by edge addition is NP-hard.

In this work, we consider a more *nimble* and in a sense *continuous* way of stabilizing a given unstable graph $G = (V, E)$. Instead of deleting/adding vertices/edges, we consider adding a small *subsidy* to a carefully chosen subset of the edges in order to create a stable weighted graph. The subsidy should be thought of as an additional incentive deployed by a central authority in order to achieve stability. A natural goal for the central authority would then be to minimize the total subsidy doled out in the stabilization process.

Definition 1 (Minimum Fractional Additive Stabilizer). Given an undirected graph $G = (V, E)$ with unit edge weights, a *fractional additive stabilizer* is a vector $c \in \mathbb{R}_+^E$ such that $(G, \mathbb{1} + c)$ is stable. In the *minimum fractional additive stabilizer* (MFASP) problem, the goal is to find a fractional stabilizer of smallest weight $\mathbb{1}^T c$.

We emphasize that we do not allow the addition of edges in MFASP, but are restricted to add weight to existing edges. We further note that the weight increases in MFASP need not be integral, and can take on arbitrary non-negative rational values.

1.1 Our contributions.

Several variants of graph stabilization are known to be NP-hard. Hardness of approximation results so far have been rather weak, however, and the gap between them and the known positive results are large. In this work, we show strong approximation-hardness results, and nearly matching positive results.

Theorem 1. *A polynomial time approximation algorithm with approximation factor $O(|V|^{1/24-\epsilon})$ where $\epsilon > 0$ for MFASP would lead to a polynomial time $O(|V|^{1/4-6\epsilon})$ -approximation for Densest k -Subgraph (DkS). Furthermore, there is no $o(\log(|V|))$ -approximation algorithm for MFASP unless $P = NP$.*

DkS is known not to possess a polynomial-time approximation scheme, assuming $NP \not\subseteq \cap_{\epsilon>0} BPTIME(2^{n^\epsilon})$ [21]. On the other hand, the best known performance guarantee of any approximation algorithm is only $\approx O(|V|^{1/4})$ [7]. It is widely believed, however, that the true approximability of DkS lies closer to the upper bound than to the hardness lower-bound. An approximation algorithm for MFASP with performance ratio significantly lower than $|V|^{1/24}$ would therefore (at the very least) be unexpected.

It is well-known that (P) has an integral solution iff the set of *inessential* vertices X (those vertices that are exposed by a maximum matching) forms an independent set (e.g., see [4, 32]). Let Y be the set of neighbours of X in G , and $Z = V \setminus (X \cup Y)$. The triple (X, Y, Z) is called the *Gallai-Edmonds decomposition* of G [15, 16, 17].

As we will see later, the optimization problem given by an instance of MFASP naturally decomposes into two subproblems: that of picking a maximum matching between the vertices in Y and the factor critical components in $G[X]$, and that of picking a maximum matching in each of the components of $G[X]$. Our two hardness results in Theorem 1 demonstrate the hardness of each of these subproblems.

In the following positive result, we let OPT denote the optimum stabilization cost of the given instance.

Theorem 2. *Let $G = (V, E)$ be a graph with Gallai-Edmonds decomposition (X, Y, Z) . If all factor critical components of $G[X]$ have size greater than one then there is a $\min\{OPT, \sqrt{|V|}\}$ -approximation algorithm for MFASP in G .*

We note that the instances generated in the hardness proofs of Theorem 1 satisfy the properties needed in Theorem 2.

While stabilization by min-edge deletion is already NP-hard in factor-critical graphs [9], we give a polynomial time algorithm to solve MFASP in factor-critical graphs.

Theorem 3. *There exists a polynomial-time algorithm to solve MFASP in factor-critical graphs.*

We further exploit the efficient solvability of MFASP in factor-critical graphs to present an exact algorithm for MFASP in general graphs whose running time is exponential only in the size of the Tutte set Y . Thus, our algorithm can be viewed as a fixed parameter algorithm (e.g., see [13]) where the parameter is the size of the Tutte set.

Theorem 4. *There exists an algorithm to solve MFASP for a graph $G = (V, E)$ with Gallai-Edmonds decomposition $V = X \cup Y \cup Z$ in time $O(2^{|Y|} \text{poly}(|V|))$.*

We conclude by giving a conditional approximation algorithm that achieves a $(k+1)/2$ -approximation when the number of non-trivial factor-critical components in the Gallai-Edmonds-decomposition exceeds the size of the Tutte set by a multiplicative factor of at least $1 + 1/k$.

1.2 Further related work.

Various ways of modifying a given graph to achieve a property have been studied in the literature, but most previous works seem to consider *monotone* properties (e.g., see [2, 3]). The König-Egerváry property is monotone while, notably, graph stability is not. Most relevant to our work are the results of Mishra et al. [26], who studied the problem of finding a minimum number of edges to delete to convert a given graph $G = (V, E)$ into a KEG. Akin to stable graphs, KEGs are also significant in game theory: an instance of the *vertex cover game* has a non-empty *core* if and only if the underlying graph is a KEG [11]. Thus, in the context of game theory, their study essentially addresses the question of how to minimally modify an instance of a vertex cover game so that the core becomes non-empty. While they showed that it is NP-hard to approximate the minimum edge-deletion problem to within a factor of 2.88, they also gave an algorithm to find a KEG subgraph with at least $3/5|E|$ edges.

In recent work, Könemann et al. [23] addressed a closely related problem of finding a minimum-cardinality set of edges to remove from a graph G such that the resulting graph has a fractional vertex cover of value at most $\nu(G)$. We note that the resulting graph here may not be stable. While this problem is known to be NP-hard [8], Könemann et al. gave an efficient algorithm to find approximate solutions in sparse graphs.

2 Preliminaries

In the rest of the paper, we will only work with unit-weight graphs as input instances for MFASP. However, the results hold for uniform weights since scaling preserves stability as well as our results.

We emphasize the following fact that is implicit from our earlier discussion. A graph G is stable iff there is a maximum matching M and $y \in \mathbb{R}_+^V$ such that the characteristic vector χ_M of M and y form an optimal pair of solutions for (P) and (D). A direct consequence of complementary slackness is then that $y_v = 0$ if v is M -exposed, as well as $y_v + y_u = w_{uv}$ for all $\{u, v\} \in M$. A feasible solution to a MFASP instance $G = (V, E)$ is determined by a triple (M, y, c) , where M is a matching, y is a fractional $1 + c$ -vertex cover satisfying $\sum_{e \in M} 1 + c_e = \sum_{v \in V} y_v$. Moreover, such a matching M will be a maximum $(1 + c)$ -weight matching. Note that we use w_e to refer to the total edge weight of an edge e , while c_e to refer to the weight added for stabilizing.

We recall the following properties of the Gallai-Edmonds decomposition (as defined in Section 1.1) (e.g., see [24, 29]): Let $G = (V, E)$ and $V = X \cup Y \cup Z$ be the Gallai-Edmonds decomposition of G . Then

- (i) every maximum matching in G contains a perfect matching in $G[Z]$,
- (ii) every connected component in $G[X]$ is factor-critical,
- (iii) every maximum matching exposes at most one vertex in every connected component of $G[X]$, and
- (iv) every maximum matching matches the vertices in Y to distinct components of $G[X]$.

We say that a component in $G[X]$ is non-trivial if it contains more than one vertex.

3 Structural Results

In this section, we show structural properties of optimal solutions to MFASP which are useful to show hardness and design algorithms. The properties are summarized in the theorem below.

Theorem 5. *Let $G = (V, E)$ be an instance of MFASP. Then,*

- (i) *for every optimal solution (M^*, y^*, c^*) ,*
 - (a) *$c_e^* = 0$ for all edges $e \in E \setminus M^*$, $0 \leq c_e^* \leq 1$ for all edges $e \in M^*$, and*
 - (b) *$|M^*| = \nu(G)$, i.e., M^* is a maximum cardinality matching in G .*
- (ii) *there exists an optimal solution (M^*, y^*, c^*) of MFASP with*
 - (c) *half-integral c^* , and*
 - (d) *$y^* \in \{0, 1/2, 1\}^{|V|}$ with the support of y^* containing the Tutte set.*

In the context of network bargaining games, the above structural theorem (property (i)(b)) tells us that there exists a way to stabilize through a minimum fractional additive stabilizer without changing *the number of deals* in the instance.

We split the proof of Theorem 5 into several Lemmas. Lemma 6 proves property (i)(a), Lemma 7 proves property (i)(b) and Lemmas 8 and 10 prove property (ii).

Lemma 6. *Let c be a minimum fractional additive stabilizer for a graph G . Let M be a matching of maximum $(\mathbb{1} + c)$ -weight. Then $c_e = 0$ for all edges $e \in E \setminus M$ and $0 \leq c_e \leq 1$ for all edges $e \in M$.*

Proof. Let y be a minimal fractional $(\mathbb{1} + c)$ -vertex cover. Since c is a fractional stabilizer for G , by the discussion in Section 2, it follows that M and y satisfy complementary slackness.

Let $\{u, v\} \in E \setminus M$. If $c_{uv} > 0$, then we may decrease c_{uv} to zero: since y is still a feasible fractional $(\mathbb{1} + c)$ -vertex cover, and y satisfies complementary slackness with M , we obtain a better fractional additive stabilizer, thus contradicting the optimality of c . Thus, for every edge $e \in E \setminus M$, we have $c_e = 0$.

Let $\{u, v\} \in M$. Then by complementary slackness, we have that $y_u + y_v = 1 + c_{uv}$. If $c_{uv} > 1$, then we obtain (c', y') where $c'_{uv} := 1$, $c'_e := c_e$ for every edge $e \in E \setminus \{uv\}$ and $y'_u := 1$, $y'_v := 1$, $y'_i := y_i$ for every vertex $i \in V \setminus \{u, v\}$. The resulting solution y' is a feasible fractional $(\mathbb{1} + c')$ -vertex cover and y' satisfies complementary slackness with M . Thus, c' is a fractional additive stabilizer. We note that $\sum_{e \in E} c'_e < \sum_{e \in E} c_e$, a contradiction to the optimality of c . \square

Lemma 7. *For a graph G , let c be a minimum fractional additive stabilizer. Then, the cardinality of a maximum $(\mathbb{1} + c)$ -weight matching is equal to the maximum cardinality of a matching in G .*

Proof. Let M be a maximum $(1+c)$ -weight matching. For the sake of contradiction, suppose the cardinality of M is strictly less than the cardinality of a maximum matching in G . Let y be a minimum fractional $(1+c)$ -vertex cover. Then, M and y satisfy complementary slackness.

Since, by our assumption, M is not a maximum cardinality matching in G , there exists an M -augmenting path P . Let u_s and u_e denote the first and last vertices in the path P , respectively. Since y is a minimal fractional $(1+c)$ -vertex cover, and u_s and u_e are exposed in M , we have

$$y_u + y_v \geq 1 \quad \forall \{u, v\} \in P \setminus M, \quad (1)$$

$$y_u + y_v = 1 + c_{uv} \quad \forall \{u, v\} \in M \cap P, \quad (2)$$

$$y_{u_s} = 0 = y_{u_e}. \quad (3)$$

Let N be the matching obtained by taking the symmetric difference of M and P . Let us obtain new weights as follows:

$$c'_{uv} := \begin{cases} c_{uv} & \text{if } \{u, v\} \in E \setminus P \\ y_u + y_v - 1 & \text{if } \{u, v\} \in N \cap P \\ 0 & \text{if } \{u, v\} \in P \setminus N \end{cases}$$

We now show that the weight of matching N w.r.t. $(1+c')$ is identical to that of matching M :

$$\begin{aligned} \sum_{e \in N} (1 + c'_e) - \sum_{e \in M} (1 + c_e) &= \sum_{\{u,v\} \in N \cap P} (1 + (y_u + y_v - 1)) - \sum_{\{u,v\} \in M \cap P} (1 + c_e) \\ &= \sum_{\{u,v\} \in M \cap P} (y_u + y_v) - \sum_{\{u,v\} \in M \cap P} (1 + c_e) \\ &= \sum_{\{u,v\} \in M \cap P} (1 + c_e) - \sum_{\{u,v\} \in M \cap P} (1 + c_e) \\ &= 0. \end{aligned}$$

The second and third inequality are due to equations (3) and (2). By Definition of c' , we have that y is a feasible fractional $(1+c')$ -vertex cover in G . Moreover, by Lemma 6 and the construction of N and c' , the $(1+c')$ -weight of matching N is equal to the sum $\sum_{v \in V} y_v$ of the fractional $(1+c')$ -vertex cover y . Because of the LP duality relation between the two values, N is a matching of maximum $(1+c')$ -weight, and y is a minimum fractional $(1+c')$ -vertex cover. Hence, c' is a fractional additive stabilizer. Next we note that

$$\begin{aligned} \sum_{e \in E} c'_e - \sum_{e \in E} c_e &= \sum_{\{u,v\} \in N \cap P} (y_u + y_v - 1) - \sum_{\{u,v\} \in M \cap P} c_{uv} \\ &= \sum_{\{u,v\} \in N \cap P} (y_u + y_v) - |N \cap P| - \sum_{\{u,v\} \in M \cap P} c_{uv} \\ &= \sum_{\{u,v\} \in M \cap P} (y_u + y_v) - |N \cap P| - \sum_{\{u,v\} \in M \cap P} c_{uv} \\ &= |M \cap P| - |N \cap P| \\ &= -1. \end{aligned}$$

The third equality is due to (3), the fourth follows from (2). Hence, c' is a fractional additive stabilizer whose weight is smaller than that of c , a contradiction to the optimality of c . \square

Lemma 8. *For every graph G , there exists an optimal solution (M, y, c) of MFASP with half-integral c and half-integral y .*

Proof. Let \bar{c} be a minimum fractional additive stabilizer. By Lemma 7, we know that there exists a maximum matching in G that is also a maximum $(\mathbb{1} + \bar{c})$ -weight matching. Let M be such a matching. We consider the following linear program:

$$\begin{aligned}
\min \sum_{e \in M} c_e & & (\text{LP}(G, M)) \\
y_u + y_v = c_{uv} + 1 & & \forall \{u, v\} \in M \\
y_u + y_v \geq 1 & & \forall \{u, v\} \in E \setminus M \\
y_u = 0 & & \forall u \in V, u \text{ is exposed by } M \\
c, y \geq 0 & &
\end{aligned}$$

If (c, y) is an optimal solution of $LP(G, M)$, then c gives a minimum fractional additive stabilizer for G . In order to show that c is a fractional additive stabilizer, it is sufficient to find a fractional $(\mathbb{1} + c)$ -vertex cover y that satisfies complementary slackness conditions with M . But, by the constraints in $LP(G, M)$, it is clear that y satisfies complementary slackness conditions with M . Furthermore, c is a minimum fractional additive stabilizer, since otherwise, we could derive a contradiction to the optimality of \bar{c} . Thus, it is sufficient to show that there exists a half-integral optimal solution (c, y) of $LP(G, M)$.

We observe that if G is bipartite, then for every matching M in G , the extreme point solutions to $LP(G, M)$ are integral since the constraint matrix is totally unimodular and the right-hand side is integral.

Now, suppose $G = (V, E)$ is non-bipartite. We construct a bipartite graph $G' = (V_1 \cup V_2, E')$ as follows: for each vertex $u \in V$, we introduce vertices $u_1 \in V_1$, $u_2 \in V_2$ and for each edge $\{u, v\} \in E$, we introduce edges $\{u_1, v_2\}, \{u_2, v_1\}$ in E' . For each matching edge $\{u, v\} \in M$, we include edges $\{u_1, v_2\}, \{u_2, v_1\}$ in M' . Thus M' is a matching in G' that exposes u_1 and u_2 for every vertex $u \in V$ that is exposed by M . Let (c', y') be an integral optimal solution of $LP(G', M')$. Let (c, y) be obtained by setting $c_{uv} := 1/2(c'_{u_1 v_2} + c'_{u_2 v_1}) \forall \{u, v\} \in M$ and $y_u := 1/2(y'_{u_1} + y'_{u_2}) \forall u \in V$. Clearly, (c, y) is half-integral. The following claim shows that (c, y) is an optimum to $LP(G, M)$. \square

Claim 9. Let (c', y') be an optimal solution of $LP(G', M')$. Then (c, y) obtained by setting $c_{uv} := 1/2(c'_{u_1 v_2} + c'_{u_2 v_1})$ for all $\{u, v\} \in M$ and $y_u := 1/2(y'_{u_1} + y'_{u_2})$ for all $u \in V$ is an optimal solution for $LP(G, M)$.

Proof. The feasibility of the solution (c, y) for $LP(G, M)$ is easy to verify. We note that $\sum_{e \in M} c_e = 1/2 \sum_{e \in M'} c'_e$. We will prove optimality.

Suppose (c, y) is not optimal for $LP(G, M)$. Then there exist (\tilde{c}, \tilde{y}) feasible for $LP(G, M)$ such that $\sum_{e \in M} \tilde{c}_e < \sum_{e \in M} c_e$. Consider the solution (\tilde{c}', \tilde{y}') obtained by setting $\tilde{c}'_{u_1 v_2} = \tilde{c}'_{u_2 v_1} = \tilde{c}_{uv}$ for every $\{u, v\} \in M$ and $\tilde{y}'_{u_1} = \tilde{y}'_{u_2} = \tilde{y}_u$ for every $u \in V$. The resulting solution (\tilde{c}', \tilde{y}') is feasible to $LP(G', M')$.

Moreover $\sum_{e \in M'} \tilde{c}'_e = 2 \sum_{e \in M} \tilde{c}_e < 2 \sum_{e \in M} c_e = \sum_{e \in M'} c'_e$, a contradiction to the optimality of (c', y') . \square

Lemma 10. *For every graph G , there exists an optimal solution (M, y, c) of MFASP with half-integral c and half-integral y with the support of y containing the Tutte set.*

Proof. Let the Gallai-Edmonds decomposition of G be given by $V = X \cup Y \cup Z$. Let c be a half-integral minimum fractional additive stabilizer for G . Let M be a maximum $(1+c)$ -weight matching and y be a half-integral minimum fractional $(1+c)$ -vertex cover (such a y and c exist by Lemma 8). Suppose that $y_v = 0$ for some $v \in Y$. We will construct a half-integral fractional additive stabilizer c' without increasing the cost and a fractional $(1+c')$ -vertex cover y' that satisfies complementary slackness with M and has $y'_w > 0$ for each node w of the Tutte set.

Since M is maximum, every node of Y is matched. For $v \in Y$, we denote by S_v the factor-critical component in $G[X]$ which is matched to v and by s_v the vertex matched to v . Let $Y' := \{v \in Y : y_v = 0\}$. We set $c'_e := 0 \ \forall e \in \bigcup_{v \in Y'} (E(S_v) \cup \{v, s_v\})$ and $c'_e := c_e$ otherwise. It is clear that c' is half-integral and the cost of c' cannot be more than that of c .

We define

$$y'_w := \begin{cases} 1/2, & \text{if } w \in Y' \text{ or } w \in \bigcup_{v \in Y'} V(S_v), \\ y_w & \text{else.} \end{cases}$$

By definition, y' satisfies the covering constraints in $\tau_f(G, 1+c')$ for edges in $E[\bigcup_{v \in Y'} (S_v) \cup Y']$. For other edges $\{v, t\}$ incident to $v \in Y' \cup \bigcup_{v \in Y'} (S_v)$ either $y'_t = y_t = 1$ (if $v \in Y'$) or $y'_t \geq 1/2$ (if $t \in Y \setminus Y'$). Therefore y' is a fractional $(1+c')$ -vertex cover. Finally it is clear that y' satisfies complementary slackness with M : on every matching edge that is not adjacent to a vertex $v \in Y'$, it follows since y' takes the same values on the end vertices as y ; for a matching edge $\{u, v\}$ adjacent to a vertex $v \in Y'$, by definition of y' and c' , it follows that $y_u + y_v = 1 + c_{uv}$. \square

Using Theorem 5, we will always use and construct solutions where $y_v \geq 1/2$ for a vertex v in the Tutte set in the remaining paper. Therefore, we assume in the subsequent sections, that $Z = \emptyset$ in the Gallai-Edmonds decomposition of a graph. If that were not the case, we would first consider the graph without Z and then extend the stabilizer using a perfect matching on Z without additional cost. This is done by setting $c_e = 0$ for every edge $e = \{u, v\} \in E[Z] \cup \delta(Z)$ and $y_v = 1/2$ for all $v \in Z$.

We now use the structural insights from Theorem 5 to describe the behaviour of feasible solutions on the factor-critical components of the Gallai-Edmonds decomposition.

Lemma 11. *Let G be a graph with Gallai-Edmonds decomposition $V(G) = X \cup Y \cup Z$ and (M, y, c) be a feasible solution for MFASP in G fulfilling the properties (a), (b), (c) and (d) of Theorem 5. Let K be a non-trivial component in $G[X]$ with a vertex u such that $y_u = 0$. If K has a vertex u exposed by M , then $\sum_{e \in E(K)} c_e \geq 1$. On the other hand, if K is matched to Y by an edge $e' = \{v, w\}$ with $w \in Y$ and $y_w \geq 1/2$, then $\sum_{e \in E(K)} c_e + c_{e'} \geq y_w$.*

Proof. In this proof, we use an equivalent definition of factor-critical graphs: A graph is factor-critical if and only if it has an odd ear-decomposition. Furthermore, the initial vertex of the ear-decomposition can be chosen arbitrarily [25]. An ear-decomposition of a graph G is a sequence r, P_1, \dots, P_k with $G = (\{r\}, \emptyset) + P_1 + \dots + P_k$ such that P_i is either a path where exactly the endpoints belong to $\{r\} \cup V(P_1) \cup \dots \cup V(P_{i-1})$ or a circuit where exactly one of its vertices belongs to $\{r\} \cup V(P_1) \cup \dots \cup V(P_{i-1})$. An ear-decomposition is called odd if all P_i have odd length.

Moreover, (the proof of the above equivalence implies that) for a maximum matching M in a factor-critical graph G , an odd ear-decomposition can be chosen such that the exposed vertex is the initial vertex r , each path P_i is M -alternating such that the first and last edge are not part of M and each circuit P_i contains $|E(P_i) - 1|/2$ matching edges such that the vertex in $\{r\} \cup V(P_1) \cup \dots \cup V(P_{i-1})$ is not matched.

If u is exposed by M , then $y_u = 0$. Since K is factor-critical, there exists an M -blossom through u , i.e. a circuit C of odd length where all vertices except u are adjacent to an edge in $E(C) \cap M$ (follows from the above statement). Let the vertices of C be $u_0 = u, u_1, \dots, u_{2t}, u_{2t+1} = u$. Now consider the optimal vertex cover values $y_{u_0}, \dots, y_{u_{2t}}$ for the vertices in C . By definition of the vertex cover, $y_{u_i} + y_{u_{i+1}} \geq 1$ for every $i = 0, \dots, 2t$ and, in particular, $y_{u_1} \geq 1$ and $y_{u_{2t}} \geq 1$. Furthermore, the inequalities for the matching edges are tight and thus, $1 + c_{\{u_i, u_{i+1}\}} = y_{u_i} + y_{u_{i+1}}$ for every $i = 1, 3, 5, \dots, 2t - 1$. Therefore, summing up, we have

$$\begin{aligned} t + \sum_{e \in E(K)} c_e &\geq t + \sum_{i=1,3,5,\dots,2t-1} c_{\{u_i, u_{i+1}\}} = \sum_{i=1}^{2t} y_{u_i} \geq 1 + \left(\sum_{i=2}^{2t-1} y_{u_i} \right) + 1 \\ &= 2 + \sum_{i=2,4,\dots,2t-2} (y_{u_i} + y_{u_{i+1}}) \geq 2 + (t-1) = t+1, \end{aligned}$$

which proves the first statement.

Now, let us consider the case where K is matched. Remember that we have $y_w \in \{1/2, 1\}$. If $y_v = 0$, then the proof is identical to that of the first statement. If $y_v = 1$, clearly, $c_{e'} = y_w$. Therefore, we may assume that $y_v = 1/2$ and thus $c_{e'} = y_w - 1/2$. It remains to show that $\sum_{e \in E(K)} c_e \geq 1/2$.

Since K is factor-critical, there exists a path from u to v of odd length in K , which is M -alternating, in particular, the edges incident to u and v are non-matching edges. (The existence of such a path follows from the fact that it is possible to construct an odd ear-decomposition of K with initial vertex v such that each ear is an M -alternating path or circuit where the first and last edge are not matching edges.) Let $u = v_0, v_1, \dots, v_{2t+1} = v$ be such a path P . Suppose for the sake of contradiction $c_e = 0$ for all $e \in E(K) \supseteq P$. Since y is a vertex cover and the matching edges are tight w.r.t. y , it follows that $y_{v_i} = 1$ for odd $i \leq 2t$ and $y_{v_i} = 0$ for even i . But that implies $y_{v_{2t}} + y_v = y_v = 1/2$, a contradiction. Hence, $\sum_{e \in E(K)} c_e > 0$. Since c is half-integral, it follows that $\sum_{e \in E(P)} c_e \geq 1/2$. \square

This directly implies the following statement:

Proposition 12. *Let G be a graph with Gallai-Edmonds decomposition $V(G) = X \cup Y \cup Z$ and (M, y, c) be a (not necessarily optimum) solution for MFASP in*

G fulfilling the properties (a), (b), (c) and (d) of Theorem 5 such that $(y, c) = \operatorname{argmin}\{1^T c : (M, y, c) \text{ is feasible for MFASP}\}$. Let K be a non-trivial factor-critical component in $G[X]$. If K is matched by M , then $y_v = 1/2$ for every $v \in V(K)$ and $c(e) = 0$ for every $e \in K$.

4 Inapproximability

In this section, we will show that MFASP is hard to approximate in general graphs. We show the first part of Theorem 1 in Section 4.1 and the second part in Section 4.2.

4.1 Reduction from Densest k -Subgraph

In this subsection, we show that MFASP is at least as hard as the Densest k -Subgraph Problem in a certain approximation preserving sense. In Theorem 13, we show that a polynomial time f -approximation algorithm for the MFASP would imply a polynomial time $2f$ -approximation for the Minimum k -Edge Coverage Problem. We show that Theorem 13 implies a strong relation to the Densest k -Subgraph Problem at the end of the subsection.

We recall the two problems of relevance: Given a graph and a parameter k , the Minimum k -Edge Coverage Problem (MkEC) is to find a minimum number of vertices whose induced subgraph has at least k edges. The Densest k -Subgraph Problem asks for k vertices with a maximum number of induced edges.

Theorem 13. *If there is a polynomial time f -approximation algorithm for the MFASP, then there is a polynomial time $2f$ -approximation for MkEC.*

Proof. Let $G = (V, E)$ and $k < |E|$ be an instance of MkEC. Our goal is to find a subset of k edges spanning a minimum number of nodes. We construct an instance \hat{G} of MFASP whose Gallai-Edmonds decomposition (GED) has a specific form. By Theorem 5, it is sufficient to consider maximum matchings for MFASP. \hat{G} will encode the problem of picking k edges for MkEC as the problem of identifying k factor-critical components in the GED that are to be exposed by the matching in a solution for MFASP. An illustration can be found in Figure 1.

Let Y' be a copy of vertex set V ; we will later show that Y' is a part of the Tutte set Y of the GED of the constructed graph. Furthermore, for each edge $e = \{v, w\} \in E$, we add a triangle, and we let Δ denote the collection of these triangles. We will later show that each triangle will form a component of X in the GED. We connect each node of a triangle corresponding to an edge $\{v, w\} \in E$ to the vertices v and w in Y' . As any maximum matching matches each vertex of the Tutte set to a distinct factor-critical component, we modify the instance such that the number of triangles is exactly $|V| + k$. To achieve this, we either add vertices to Y' which are connected to all vertices of all triangles or we add triangles that are connected to all vertices in Y' .

While MkEC allows choosing any collection of k edges, there may exist a collection of k triangles in our current graph such that the remaining triangles cannot be matched perfectly to Y' . To remedy the situation, we add $q - 1$ copies Y_2, \dots, Y_q of Y' , where q will be chosen later, and connect each vertex of Y_i

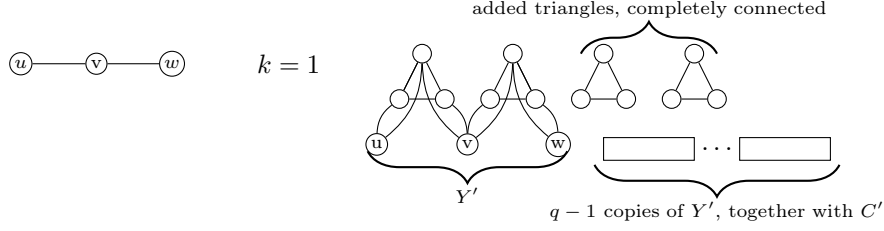


Figure 1: Instance of $MkEC$ and schema of a corresponding MFASP instance \hat{G}

($i \in 2, \dots, q$) with the same nodes as the corresponding node in the “original” set Y' , which can be seen as Y_1 . We will later show that all these copies belong to the Tutte set Y of the Gallai-Edmonds decomposition. Moreover, we add $|Y'| \cdot (q-1)$ triangles and connect all their vertices to all vertices in $Y'' := Y_1 \cup \dots \cup Y_q$. Call this set of newly added triangles C' .

The following two lemmas describe the relevant structure of the construction.

Lemma 14. *Let $q \geq \max_{v \in V(G)} |\delta(v)|$. Then for any choice of k triangles, there is a perfect matching between Y'' and the triangles that were not chosen.*

Proof. Let \mathcal{E} be the set of k triangles that we wish to expose. We construct a matching $M_{\mathcal{E}}$ that exposes precisely \mathcal{E} . For each triangle in $\Delta \setminus \mathcal{E}$ corresponding to some edge $\{u, v\} \in E$ match the triangle to a currently exposed copy of u . Note that q is at least the maximum degree in G , and hence this process matches all triangles in $\Delta \setminus \mathcal{E}$.

Let $\tilde{\Delta}$ be the collection of triangles not in $\Delta \cup \mathcal{E}$, and let \tilde{Y}'' be the collection of $M_{\mathcal{E}}$ exposed vertices in Y'' . Clearly, $|\tilde{\Delta}| = |\tilde{Y}''|$, and the graph induced by the edges between \tilde{Y}'' and the vertices of triangles in $\tilde{\Delta}$ is complete bipartite. Thus, picking any \tilde{Y}'' -perfect matching in this graph and adding its edges to $M_{\mathcal{E}}$ yields the desired matching exposing \mathcal{E} . \square

Lemma 15. *The Gallai-Edmonds decomposition of \hat{G} is given by $Y = Y''$, $X = V(\hat{G}) \setminus Y''$, $Z = \emptyset$.*

Proof. In any graph $G = (V, E)$, the size of a maximum matching can be characterized by the *Tutte-Berge formula* [6]:

$$2\nu(G) = |V| - \max_{W \subseteq V} (q_G(W) - |W|)$$

where $q_G(W)$ denotes the number of components with an odd number of nodes in $G[V \setminus W]$.

By plugging $W = Y''$ into the Tutte-Berge formula, we see that a maximum matching has size at most $\frac{V(\hat{G}) - k}{2} = 2|Y''| + k$. A matching of size $2|Y''| + k$ exists by Lemma 14.

By Lemma 14, every vertex $v \in V(\hat{G}) \setminus Y''$ is inessential, that is, there exists a maximum matching in \hat{G} exposing v . Now, suppose there was a maximum matching M exposing a vertex $v \in Y''$. We know $|M| = 2|Y''| + k$, but any matching can contain at most $|Y''| + k$ edges of $E[V(\hat{G}) \setminus (Y'')]$, as that is the number of triangles. All other edges have one endpoint in Y'' . Thus, if M exposes v , then $|M| < |Y''| + k + |Y''|$, which is a contradiction. \square

Together, Lemmas 14 and 15 imply that for any choice of k unmatched factor-critical components, there is a maximum matching exposing exactly one vertex in these k components and conversely, every maximum matching is of this form. We have shown in Theorem 5 that it suffices to consider stabilizers (M, y, c) where M is a maximum matching, y and c are half-integral and y is positive on the Tutte set. Once the set of unmatched components is fixed, we can see how to obtain an optimal stabilizer for this situation: Start with a matching between the matched components and Y'' and extend it arbitrarily to a maximum matching M . Let $K \subset V(\hat{G})$ be the set of vertices in triangles not matched to Y'' . We set $c_e = 1$ for the matching edge in each of these triangles, $y_v = 1$ for both matched vertices within the triangle and $y_v = 0$ for the remaining vertices in K . For $v \in Y''$, we set $y_v = 1$ if $v \in N(K)$ and $y_v = 1/2$ otherwise. By Proposition 12, $y_v = 1/2$ is then optimal for all vertices v in matched triangles. Consequently, $c(e) = 1/2$ if $e \in M \cap \delta(N(K))$ and $c(e) = 0$ for the remaining edges. In total, $\mathbb{1}^T c = k + 1/2|N(K)|$.

If the unmatched components correspond to a set E' of edges in G , then $N(K) \cap Y_1$ corresponds exactly to the vertices in G spanned by E' . Consequently, the cost of the stabilizer consists of k (for the unmatched triangles) and q times the number of spanned vertices in G as the neighbourhoods of the copies of Y' are identical. Suppose a component that does not correspond to an edge in G is unmatched. Then, $y_v = 1$ for all $v \in Y''$ and therefore $c(E) = k + 1/2|Y''|$. Thus, we can modify the solution by choosing to expose components corresponding to edges instead without increasing the cost. W.l.o.g. we modify any solution to MFASP this way. Then, we have the following Lemma:

Lemma 16. *G has a solution of MkEC of size at most x if and only if \hat{G} has a MFASP of cost at most $k + qx/2$.*

We next show that Lemma 16 yields a factor-preserving hardness. If $k > 0$, then we have $x \geq 2$. Moreover, set $q = \max\{k, \max_{v \in G} |\delta(v)|\}$. Let x^* be the value of an optimal solution for MkEC, then the optimal value of MFASP is $k + \frac{qx^*}{2}$. Suppose there was an f -approximation for MFASP, this would yield a stabilizer solution of cost $k + \frac{qx}{2} \leq f(k + \frac{qx^*}{2})$ for some x . We observe that $\frac{qx}{2} \leq (f-1)k + f\frac{qx^*}{2} \leq qf(1 + \frac{x^*}{2}) \leq qfx^*$. Therefore, we have a $2f$ -approximate solution of MkEC which proves Theorem 13. \square

We now show the first part of Theorem 1. While using Theorem 13 to derive hardness of approximation for MFASP, we have to be careful if we want to set f to be a value that depends on the input size: If G is the input for MkEC with $n = |V(G)|$, then the number of vertices in our construction of the MFASP is bounded by $\max\{4n^3 + n^2, 7n^2\} \leq 7n^3$. For \hat{n} being the number of vertices in an MFASP instance, we can conclude that an approximation algorithm with approximation factor $O(\hat{n}^{\frac{1}{24}-\epsilon})$, for $\epsilon > 0$ would lead to a $O(n^{1/8-3\epsilon})$ approximation algorithm for MkEC, where n is the number of vertices of the MkEC instance. This would lead to an algorithm with approximation factor $O(n^{1/4-6\epsilon})$ for MkDS according to [18].

4.2 Reduction from Set Cover

While the Densest- k -subgraph problem is believed to be difficult, there are no strong inapproximability results known. In this subsection, we show Set-Cover-

hardness for MFASP, which leads to a stronger inapproximability result.

We exploit a different aspect of MFASP for this reduction: We could look at MFASP as a problem consisting of two subproblems: How to choose the matched factor-critical components and, having fixed those, how to choose the matching within the unmatched components and thus decide the y -values. In the previous reduction, the difficulty was completely encoded in the first subproblem. Once we chose the matched components, the second problem was easy. In the following reduction, we will consider a construction, where the matched components are the same for any reasonably good solution and the difficulty lies in the second subproblem.

Theorem 17. *If there is a polynomial time f -approximation algorithm for MFASP, then there is a polynomial time $2f$ -approximation algorithm for Set Cover.*

Proof. Let $(\mathcal{S}, \mathcal{X})$ be an instance of the Set Cover problem with sets $\mathcal{S} = \{S_1, \dots, S_m\}$ and elements $\mathcal{X} = \{x_1, \dots, x_n\}$. Our goal is to choose a minimum number of sets whose union contains all elements x_i . Let the frequency of element x_i be $F_i = |\{S_j : x_i \in S_j\}|$. Without loss of generality, $F_i > 1$. Otherwise, the only set containing an item x_i has to be part of any solution, so it suffices to consider instances with $F_i > 1$ for all $i \in [n]$.

We construct a graph \hat{G} with a specific Gallai-Edmonds decomposition. Our goal will be to decide whether a set is included in a set cover based on the y -values of the Tutte set. For each set S_j , create n vertices S_j^1, \dots, S_j^n . Let $Y' = \{S_j^i : i \in [n], j \in [m]\}$. This will be our Tutte set. For each S_j^i create a clique C_j^i of size $2N+1$ with $N > (nm)^2$ with a designated vertex c_j^i and add an edge $\{S_j^i, c_j^i\}$. The purpose of these large cliques is ensuring that every vertex in Y' is matched to its clique, thus exposing the factor-critical components we construct next: For each element x_i with F_i odd, construct an odd cycle Q_i consisting of F_i vertices $x_i^1, \dots, x_i^{F_i}$. For each element with F_i even, construct an odd cycle Q_i consisting of $F_i + 1$ vertices $x_i^1, \dots, x_i^{F_i+1}$, where the vertex $x_i^{F_i+1}$ is a dummy vertex. Let $\hat{S}_{(1,i)}, \dots, \hat{S}_{(F_i,i)}$ denote the sets in \mathcal{S} containing x_i (choose the order arbitrarily). Consider the n copies of the corresponding vertices in Y' and add edges $\{x_i^k, \hat{S}_{(k,i)}^\ell\} \forall \ell \in [n], \forall k \in [F_i] \forall i \in [n]$. I.e. add an edge between the k -th vertex for x_i and all copies of the k -th set in the list. For every $i \in [n]$ with F_i even, add edges between $x_i^{F_i+1}$ and all vertices in Y' . Let the resulting graph be $\hat{G} = (\hat{V}, \hat{E})$. (See figure 2).

We now analyze the structure of the instance we built:

Claim 18. The Gallai-Edmonds decomposition of G is $X \cup Y \cup Z$ where $Z = \emptyset$, $Y = Y'$ and $X = \hat{V} \setminus Y'$.

Proof. Using the Tutte-Berge formula for the set $W = \{S_j^i : 1 \leq j \leq m, 1 \leq i \leq n\}$, we see that a maximum matching has size at most $nm(N+1) + \sum_{i=1}^n \lfloor F_i/2 \rfloor$. Clearly, a matching of this size exposing a vertex v can be constructed for any $v \in C_j^i$ or $v \in Q_i$. Moreover, a matching of this size cannot be constructed by exposing a vertex $v = S_j^i$: If so, such a vertex S_j^i would belong to a factor-critical component K in the Gallai-Edmonds decomposition and K also contains C_j^i . But factor-critical graphs are 2-edge connected, and removing the edge (S_j^i, c_j^i) would separate the graph K into two components, a contradiction. \square

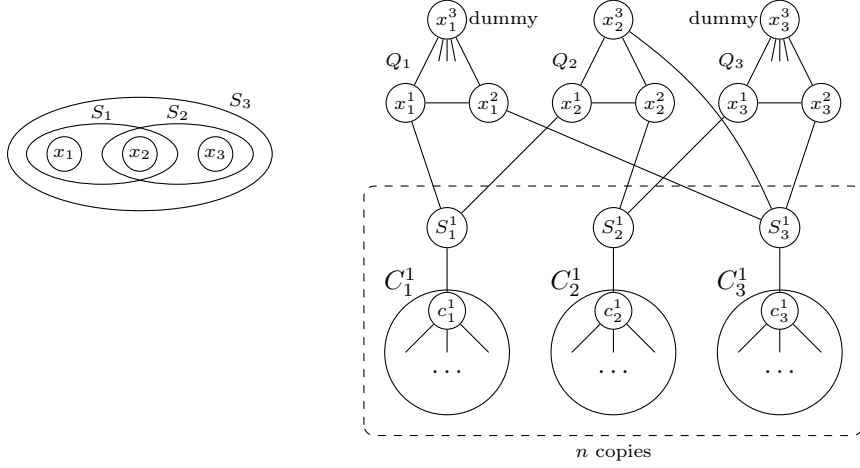


Figure 2: Set Cover instance and constructed MFASP instance \hat{G}

Claim 19. Let $T \subseteq [m]$ denote the indices corresponding to a set cover. Then there exists a feasible solution to the MFASP instance \hat{G} whose stabilizer cost is at most $n(1 + |T|/2)$.

Proof. For each $i \in [n]$, let k_i denote an arbitrarily chosen index in T such that the set S_{k_i} contains the element x_i .

Consider a matching \bar{M} obtained by matching S_j^i with $c_j^i \forall j \in [m], i \in [n]$, picking a perfect matching of the rest of the clique vertices $V(C_j^i) \setminus \{c_j^i\}$, exposing $x_i^{k_i}$ and picking a perfect matching of the rest of the vertices in each odd cycle Q_i .

Obtain a fractional vertex cover solution \bar{y} as follows: For every $i \in [n]$, let $\bar{y}_{S_j^i} = 1$ if $j \in T$ and $\bar{y}_{S_j^i} = 1/2$ if $j \in [m] \setminus T$. For each $i \in [n]$, set $\bar{y}_{x_{k_i}} = 0$, $\bar{y}_{x_k} = 1$ for the two vertices x_k in Q_i that are adjacent to x_{k_i} and $\bar{y}_{x_k} = 1/2$ for the other vertices in Q_i .

Obtain the solution \bar{c} as $\bar{c}_{uv} = \bar{y}_u + \bar{y}_v - 1$ for every $uv \in \bar{M}$. Then the solution $(\bar{M}, \bar{y}, \bar{c})$ is a feasible solution to the MFASP instance \hat{G} . Moreover, the cost of the stabilizer $\mathbf{1}^T \bar{c}$ is $n(1 + |T|/2)$. \square

Let (M, y, c) be an f -approximate feasible solution to the MFASP instance \hat{G} . We now can assume the following properties. If these are not fulfilled, we can change the solution without increasing the cost.

Claim 20. We can assume the following properties.

1. M matches S_j^i to c_j^i for every $j \in [m]$, $i \in [n]$ and $y_v = 1/2$ for every $v \in V(C_j^i)$.
2. $y_{S_j^i} = y_{S_j^1}$ for every $i \in [n]$, $j \in [m]$.

Proof. We split the proof into two parts and show both properties separately.

1. If this was not the case, then there is at least one clique C_j^i with a vertex v with $y_v = 0$. Thus, $y_w = 1$ for all $w \in V(C_j^i) \setminus v$. By the complementary

slackness condition given in Section 2, we have $y_j + y_k = 1 + c_{jk}$ for each matching edge $\{j, k\} \in M$. Thus, $\sum_{e \in M \cap E(C_j^i)} c_e = |M \cap E(C_j^i)| = N$. However, we note that $T = [m]$ is a feasible set cover and by Claim 19, this gives a feasible stabilizer of cost at most $nm/2 + n$.

2. If $y_{S_j^i} \neq y_{S_j^1}$, then consider the block i_0 such that $\sum_{j \in [m]} y_{S_j^{i_0}}$ is minimum. Since the neighborhood of $\{S_1^{i_0}, \dots, S_m^{i_0}\}$ is identical to the neighborhood of all other blocks, we may copy the same vertex cover values y for all other blocks and obtain a stabilizer with non-increasing cost.

□

Therefore, the set of M -exposed vertices contains exactly one vertex in each odd cycle Q_i . Moreover, we can assume an exposed vertex is not a dummy vertex $x_i^{F_i+1}$. Otherwise, we could change that without increasing the cost of the stabilizer.

Claim 21. Let X be the M -exposed vertices. Let $P := \{j \in [m] : S_j^1 \in N(X)\}$. Then $\{S_j : j \in P\}$ is a set cover of cardinality at most $2f|P^*|$, where P^* is the set of indices corresponding to the optimal set cover.

Proof. We first show that P is indeed a set cover. Consider an element x_i . By Claim 20, there exists $k \in [F_i]$ such that x_i^k is M -exposed. There exists a set S_r^1 adjacent to x_i^k . Hence $r \in P$ and thus S_r covers x_i .

It remains to bound the cardinality of P . Since M exposes exactly one vertex in each odd cycle Q_i , by complementary slackness conditions, $\sum_{e \in Q_i} c_e \geq 1$. Thus, $\sum_{i \in [n]} \sum_{e \in Q_i} c_e \geq n$. Let $r \in P$. So S_r^1 is adjacent to an M -exposed vertex in Q_i and hence $y_{S_r^1} \geq 1$. By Claim 20, we have that $y_{S_r^i} \geq 1$ for every $i \in [n]$. Since $y_v = 1/2$ for every $v \in V(C_j^i)$ (using Proposition 12), we have that $c_{S_r^i, c_r^i} \geq 1/2$. Thus, $\sum_{r \in P} \sum_{i \in [n]} c_{S_r^i, c_r^i} \geq |P|n/2$. Thus, the cost of the stabilizer $\mathbb{1}^T c \geq n(1 + |P|/2)$. Hence, $|P| \leq 2(\mathbb{1}^T c/n - 1) \leq 2(f\mathbb{1}^T c^*/n - 1)$.

By Claim 19, we have that the cost of the optimal stabilizer $\mathbb{1}^T c^*$ is at most $n(1 + |P^*|/2)$. Thus, $|P| \leq f|P^*| + 2(f - 1) \leq 2f|P^*|$. □

Theorem 17 follows by Claim 21. □

We now show that Theorem 17 implies the second part of Theorem 1. Note that Dinur and Steurer [12] showed that there is no $(\log(n) - \epsilon)$ -approximation algorithm for Set Cover, even if the number of sets m is at most n^2 , unless $P=NP$. Theorem 17 implies that there is no $1/2(\log(n) - \epsilon)$ -approximation for MFASP, where n is the number of elements of the corresponding Set Cover instance. Now let \hat{n} denote the number of vertices in the MFASP instance constructed in the proof of Theorem 17. We have

$$\hat{n} \leq nm(2(nm)^2 + 3) + nm \leq 6(nm)^3 \leq 6n^9.$$

Hence, for \hat{n} being the number of vertices in an MFASP instance, we conclude that unless $P=NP$, there is no approximation algorithm for MFASP with approximation factor better than $(1/20)\log(\hat{n}) - \epsilon$.

5 An OPT-approximation in graphs with no singletons

In this section, we present an algorithm that achieves a $\min\{\sqrt{n}, OPT\}$ -approximation factor in graphs whose Gallai-Edmonds decomposition has no trivial factor critical components. As a subroutine, we use an extension of an algorithm to solve MFASP in factor-critical graphs. We mention how to do this by solving an LP. Note that this is also possible using “combinatorial techniques” (in particular, without solving an LP) by computing a certain minimum vertex cover in a constructed bipartite graph. However, we will not go into details here.

Our main theorem is the following:

Theorem 2. *Let $G = (V, E)$ be a graph with Gallai-Edmonds decomposition (X, Y, Z) . If all factor critical components of $G[X]$ have size greater than one then there is a $\min\{OPT, \sqrt{|V|}\}$ -approximation algorithm for MFASP in G .*

In the remainder of this Section, we prove Theorem 2. We describe the algorithm as part of the proof, but for an overview, we also give the pseudocode at the end of this section. Fix an optimum solution (M^*, y^*, c^*) satisfying the properties (c) and (d) given in Theorem 5 and Lemma 10. Then, by Proposition 12, $c_e^* = 0$ for e in a component that is not M^* -exposed. Moreover, as mentioned before, w.l.o.g. $Z = \emptyset$. As usual, $OPT := \sum_{e \in E} c^*(e)$. Let r denote the difference between the number of components in $G[X]$ and the number of vertices in Y . As M^* is a maximum matching, the properties of the Gallai-Edmonds decomposition imply that M^* exposes exactly r vertices, at most one in each component of $G[X]$. Further, M^* matches at most one vertex of a component to a vertex in Y , while the rest are matched within the component.

For each factor-critical component K , we compute a lower bound on the cost of an optimum stabilizer where the matching exposes K .

Lemma 22. *Let K be a (non-trivial) factor-critical component in $G[X]$. For each vertex w in K , let $\ell_{K,w}$ denote the optimum value of the following LP:*

$$\begin{aligned} \ell_{K,w} := \min \quad & \sum_{v \in V(K) \cup N_G(V(K))} y_v - \left(\frac{|V(K)| - 1}{2} \right) - \frac{|N_G(V(K))|}{2} \\ \text{s.t. } & y_i + y_j \geq 1 \quad \forall \{i, j\} \in E[V(K) \cup N_G(V(K))] \\ & y_i \geq 1/2 \quad \forall i \in N_G(V(K)) \\ & y_i \geq 0 \quad \forall i \in V(K) \\ & y_w = 0 \end{aligned}$$

Let $f(K) := \min_{w \in V(K)} \ell_{K,w}$. If M^* exposes a vertex in K then $\mathbb{1}^T c^* \geq f(K) + r - 1$.

We now show that Lemma 22 can be used to obtain an optimal solution for MFASP in factor-critical graphs and thereby prove Theorem 3. We note that factor-critical graphs are the special case where G consists of one component K (and thus $N(V(K)) = \emptyset$). In that case an optimum stabilizer can be obtained by computing $f(K)$, choosing any matching M^* exposing $w^* = \operatorname{argmin}_{w \in V(K)} \ell_{K,w}$ and setting c^* to fulfill complementary slackness (i.e., if y^* is a solution for ℓ_{K,w^*} , then set $c^*(uv) := y_u^* + y_v^* - 1$ for every $uv \in M^*$ and $c^*(uv) = 0$ for every $uv \in E \setminus M^*$).

Proof of Lemma 22. Recall that (M^*, y^*, c^*) is an optimum MFASP solution that satisfies the properties (c) and (d) of Theorem 5. We then have

$$\mathbb{1}^T c^* = \sum_{e \in M^*} c_e^* = \sum_{K' \neq K: K' \text{ is } M^* \text{-exposed}} \sum_{e \in M^* \cap E(K')} c_e^* + \sum_{e \in M^* \cap E(K)} c_e^* + \sum_{e \in M^* \cap \delta_G(Y)} c_e^*.$$

The first double-sum on the right-hand side is at least $r - 1$ by Lemma 11. (This Lemma only can be applied, because the factor-critical components are non-trivial.) By complementary slackness conditions as mentioned in Section 2, we know that for every edge $\{i, j\} \in M^*$, we have $1 + c_{ij}^* = y_i^* + y_j^*$. As M^* exposes one vertex in K ,

$$\sum_{e \in M^* \cap E(K)} c_e^* = \sum_{\{i, j\} \in M^* \cap E(K)} (y_i^* + y_j^* - 1) = \sum_{v \in V(K)} y_v^* - \left(\frac{|V(K)| - 1}{2} \right).$$

If $\{i, j\} \in M^*$ with $i \in Y$, then $j \in X$ is a vertex in a factor-critical component that is matched by M^* . By Proposition 12, we have that $y_j^* = 1/2$. Hence,

$$\begin{aligned} \sum_{e \in M^* \cap \delta_G(Y)} c_e^* &= \sum_{\{i, j\} \in M^*: i \in Y, j \in X} (y_i^* + y_j^* - 1) \\ &= \sum_{i \in Y} \left(y_i^* - \frac{1}{2} \right) \geq \sum_{i \in N_G(V(K))} y_i^* - \frac{|N_G(V(K))|}{2}. \end{aligned}$$

Let w be a M^* -exposed vertex in K . Then, y^* restricted to the vertices $V(K) \cup N_G(V(K))$ is a feasible solution to the LP corresponding to $\ell_{K, w}$. Combining the three relations, we get that

$$\begin{aligned} \sum_{e \in M^*} c_e^* &\geq r - 1 + \sum_{v \in V(K) \cup N_G(V(K))} y_v^* - \left(\frac{|V(K)| - 1}{2} \right) - \frac{|N_G(V(K))|}{2} \\ &\geq r - 1 + \ell_{K, w} \geq r - 1 + f(K). \end{aligned}$$

□

In order to identify a suitable matching to stabilize, we build an auxiliary graph G' as follows: Contract each component K in $G[X]$ to a pseudo-vertex v_K and assign edge weight $w_e := f(K)$ for all edges e incident to the contracted vertex v_K . Compute a matching M in G' of maximum weight covering Y .

Lemma 23. *The cost $\mathbb{1}^T c^*$ of an optimum stabilizer (M^*, c^*, y^*) is at least*

$$r - 1 + \max_{K: v_K \text{ is } M\text{-exposed}} f(K).$$

Proof. Let $K = \operatorname{argmax}_{K: v_K \text{ is } M\text{-exposed}} f(K)$. If M^* exposes K , then Lemma 22 proves the claim. So, we may assume that M^* matches K . Consider M^* restricted to the edges in the bipartite graph G' . Both M and M^* are maximum cardinality matchings in G' and v_K is M -exposed. So, we have an M -alternating path P starting from v_K and ending at another vertex corresponding to a contracted factor-critical component. Let $P = v_{K_1}, b_1, v_{K_2}, b_2, \dots, v_{K_{t-1}}, b_{t-1}, v_{K_t}$ for some $t \geq 1$ and $v_{K_1} = v_K$ and v_{K_t} is M^* -exposed. Since M is a maximum weight matching, we have $\sum_{e \in M \Delta P} w_e \leq \sum_{e \in M \cap P} w_e$. Thus, $\sum_{i=1}^{t-1} f(K_i) \leq \sum_{i=2}^t f(K_i)$ and we have that $f(K) = f(K_1) \leq f(K_t)$. Thus, the cost of the stabilizer c^* is at least $r - 1 + f(K_t) \geq r - 1 + f(K)$. □

We now stabilize M . For each M -exposed vertex v_K , let

$$w_K := \operatorname{argmin}_{w \in V(K)} \ell_{K,w},$$

and let \overline{y}^{w_K} denote the solution y achieving the optimum for ℓ_{K,w_K} . Extend M inside each factor-critical component K : if v_K is matched by M using edge $\{u, b\}$ where $u \in V(K)$, $b \in Y$, then extend M using a matching in K that exposes u . If v_K is exposed by M , extend M using a matching in K that exposes w_K . Let \overline{M} denote the resulting matching.

For each vertex v_K matched by M , set $\overline{y}_u = 1/2$ for all vertices $u \in V(K)$. For each vertex v_K that is exposed by M , set $\overline{y}_u = \overline{y}_u^{w_K}$ for all vertices $u \in V(K)$. For each vertex $b \in Y$ that is adjacent to a M -exposed v_K , set $\overline{y}_b = \max_{K: v_K \text{ is } M\text{-exposed}} \overline{y}_b^{w_K}$. For each vertex $b \in Y$ with no adjacent M -exposed v_K , set $\overline{y}_b = 1/2$. Note that these are only good choices because no trivial factor-critical components exist. For trivial components, there are cases where (for any reasonably good solution) even though the trivial component is matched, its y -value must be 0.

Set $\overline{c}(uv) = \overline{y}_u + \overline{y}_v - 1$ for edges $\{u, v\} \in \overline{M}$ and $\overline{c}(uv) = 0$ for edges $\{u, v\} \in E \setminus \overline{M}$.

We next show that the solution $(\overline{M}, \overline{y}, \overline{c})$ is a feasible solution.

Lemma 24. *$(\overline{M}, \overline{y}, \overline{c})$ is a feasible solution to MFASP.*

Proof. By construction, \overline{M} is a matching and $\sum_{e \in \overline{M}} (1 + \overline{c}_e) = \sum_{e \in \overline{M}} (\overline{y}_u + \overline{y}_v)$. It remains to show that \overline{y} is a feasible fractional w -vertex cover for $w_e = 1 + \overline{c}_e$ for every $e \in E$.

Consider an edge $e = \{u, v\} \in E$. If $e \in \overline{M}$, then $\overline{y}_u + \overline{y}_v = 1 + \overline{c}_{uv}$. Let $e \in E \setminus \overline{M}$. For such edges, we have $\overline{c}_e = 0$ and hence $1 + \overline{c}_e = 1$.

We distinguish several cases. If $e \in K$ where v_K is matched by M , then $\overline{y}_u = \overline{y}_v = 1/2$ and hence $\overline{y}_u + \overline{y}_v = 1$. If $e \in K$ where v_K is exposed by M , then $\overline{y}_u + \overline{y}_v = \overline{y}_u^{w_K} + \overline{y}_v^{w_K} \geq 1$ by the feasibility of the solution \overline{y}^{w_K} to the LP corresponding to $\ell_{K,w}$. If $e \in \delta_G(Y)$, then let $u \in Y, v \in V(K)$. If $v \in V(K)$ where v_K is matched by M , then $\overline{y}_v = 1/2$ and moreover $\overline{y}_u \geq 1/2$ and hence $\overline{y}_u + \overline{y}_v \geq 1$. If $v \in V(K)$ where v_K is exposed by M , then $\overline{y}_v = \overline{y}_v^{w_K}$ and $\overline{y}_u = \max_{w_K: v_K \text{ is } M\text{-exposed}} \overline{y}_u^{w_K} \geq \overline{y}_u^{w_K}$. By the feasibility of the solution \overline{y}^{w_K} to the LP corresponding to $\ell_{K,w}$, we have that $\overline{y}_u + \overline{y}_v \geq \overline{y}_u^{w_K} + \overline{y}_v^{w_K} \geq 1$. \square

We now bound the cost of the constructed solution $(\overline{M}, \overline{y}, \overline{c})$.

Lemma 25. *The cost $\mathbb{1}^T \overline{c}$ of the stabilizer $(\overline{M}, \overline{y}, \overline{c})$ is at most $(\sum_{e \in E} c_e^*)^2$.*

Proof. Let \mathcal{K} be the set of components such that v_K is M -exposed. The cost of $(\overline{M}, \overline{y}, \overline{c})$ is

$$\sum_{e \in \overline{M}} \overline{c}_e = \sum_{K \in \mathcal{K}} \sum_{\{u, v\} \in \overline{M} \cap K} (\overline{y}_u + \overline{y}_v - 1) + \sum_{u \in Y, v \in X: \{u, v\} \in M} (\overline{y}_u + \overline{y}_v - 1)$$

We next bound the second term in the above sum using $y_v = 1/2$ for $v \in X$ with $\{u, v\} \in M$. Let

$$Y' = \{u \in Y : u \text{ is not adjacent to an } M\text{-exposed vertex}\}.$$

For $u \in Y \setminus Y'$, we have $\bar{y}_u = 1/2$ and such vertices do not contribute to the sum.

$$\begin{aligned} \sum_{u \in Y} \left(\bar{y}_u - \frac{1}{2} \right) &\leq \sum_{u \in Y'} \left(\sum_{K \in \mathcal{K}: u \in N_G(V(K))} \left(\bar{y}_u^{w_K} - \frac{1}{2} \right) \right) \\ &= \sum_{K \in \mathcal{K}} \left(\sum_{u \in Y \cap N_G(V(K))} \bar{y}_u^{w_K} - \frac{|N_G(V(K))|}{2} \right) \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{e \in \bar{M}} \bar{c}_e &\leq \sum_{K \in \mathcal{K}} \left(\sum_{v \in V(K) \cup N_G(V(K))} \bar{y}_v^{w_K} - \frac{|V(K)| - 1}{2} - \frac{|N_G(V(K))|}{2} \right) \\ &\leq r \max_{K \in \mathcal{K}} f(K) \\ &\leq \left(\frac{r + \max_{K \in \mathcal{K}} f(K)}{2} \right)^2 \quad (\text{since arithmetic mean is at least geometric mean}) \\ &\leq \left(\frac{1 + \sum_{e \in E} c_e^*}{2} \right)^2 \quad (\text{by Lemma 23}) \\ &\leq \left(\sum_{e \in E} c_e^* \right)^2. \end{aligned}$$

□

If $OPT > \sqrt{n}$, any solution fulfilling properties (a) (b), (c) and (d) of Theorem 5 is a \sqrt{n} -approximation as the cost of any such solution is bounded by $\nu(G) \leq n/2$. Therefore, Lemmas 25 and 24 and the construction of $(\bar{M}, \bar{y}, \bar{c})$ imply Theorem 2. We give an overview of the algorithm here:

Algorithm

1. For each factor-critical component K in $G[X]$:
 - (a) For each vertex w in K , solve following LP:

$$\begin{aligned} \ell_{K,w} := \min \quad & \sum_{v \in V(K) \cup N_G(V(K))} y_v - \left(\frac{|V(K)| - 1}{2} \right) - \frac{|N_G(V(K))|}{2} \\ & y_i + y_j \geq 1 \quad \forall \{i, j\} \in E[V(K) \cup N_G(V(K))] \\ & y_i \geq 1/2 \quad \forall i \in N_G(V(K)) \\ & y_i \geq 0 \quad \forall i \in V(K) \\ & y_w = 0 \end{aligned}$$

- (b) Let $f(K) := \min_{w \in V(K)} \ell_{K,w}$.
2. Construct an auxiliary bipartite graph G' from G as follows: Contract each component K in $G[X]$ to a pseudo-vertex v_K and assign edge weight $w_e := f(K)$ for all edges e incident to the contracted vertex v_K . Delete edges in $E[Y]$.

3. Compute a matching M in G' of maximum weight covering Y .
 4. For each M -exposed vertex v_K , let $w_K := \operatorname{argmin}_{w \in V(K)} \ell_{K,w}$, let \bar{y}^{w_K} denote the solution y achieving the optimum for ℓ_{K,w_K} .
 5. Identify a matching \bar{M} : Extend M inside each factor-critical component K : if v_K is matched by M using edge u, b where $u \in V(K), b \in Y$, then extend M using a matching in K that exposes u . If v_K is exposed by M , extend M using a matching in K that exposes w_K . Let \bar{M} denote the resulting matching.
 6. Identify a fractional vertex cover \bar{y} : For each vertex v_K matched by M , set $\bar{y}_u = 1/2$ for all vertices $u \in V(K)$. For each vertex v_K that is exposed by M , set $\bar{y}_u = \bar{y}_u^{w_K}$ for all vertices $u \in V(K)$. For each vertex $b \in Y$ that is adjacent to a M -exposed v_K , set $\bar{y}_b = \max_{K: v_K \text{ is } M\text{-exposed}} \bar{y}_b^{w_K}$. For each vertex $b \in Y$ with no adjacent M -exposed v_K , set $\bar{y}_b = 1/2$.
 7. Identify a feasible MFASP solution \bar{c} : Set $\bar{c}(uv) = \bar{y}_u + \bar{y}_v - 1$ for edges $\{u, v\} \in \bar{M}$ and $\bar{c}(uv) = 0$ for edges $\{u, v\} \in E \setminus \bar{M}$.
 8. Return $(\bar{M}, \bar{y}, \bar{c})$.
-

6 An Exact Algorithm for MFASP

In this section, we describe an exact algorithm to solve the MFASP in arbitrary graphs G . The algorithm is based on the Gallai-Edmonds decomposition $V(G) = X \cup Y \cup Z$ and makes use of a polynomial time exact algorithm to solve MFASP in the factor-critical components in $G[X]$. The runtime of our algorithm grows exponentially only in the size of the Tutte set. Thus, our algorithm is fixed parameter tractable w.r.t. the size of the Tutte set Y . In particular, the resulting algorithm runs in polynomial-time if the size of the Tutte set is bounded by $\mathcal{O}(\log n)$.

Theorem 4. *There exists an algorithm to solve MFASP for a graph $G = (V, E)$ with Gallai-Edmonds decomposition $V = X \cup Y \cup Z$ in time $O(2^{|Y|} \operatorname{poly}(|V|))$.*

Outline of the algorithm. By property (ii) in Theorem 5, we know that there exists a subset $S^* \subseteq Y$ and a half-integral minimum fractional stabilizer c^* with a half-integral minimum fractional $(1 + c^*)$ -vertex cover solution y^* such that $y_v^* = 1$ for all $v \in S^*$, and $y_v^* = \frac{1}{2}$ for all $v \in Y \setminus S^*$. This motivates the following problem: given a set $\hat{S} \subseteq Y$, find a minimum fractional stabilizer c which admits a minimum fractional $(1 + c)$ -vertex cover y satisfying $y_v = 1$ if $v \in \hat{S}$ and $y_v = \frac{1}{2}$ if $v \in Y \setminus \hat{S}$. Or, decide that no such solution exists. In Section 6.1 we present a polynomial-time algorithm for this problem. Repeatedly applying this algorithm to all subsets of the Tutte set and searching for the optimal one gives the optimal solution to MFASP and implies Theorem 4.

6.1 Algorithm to find the optimal stabilizer knowing the subset of Tutte vertices with y-value one

Let G be a graph with Gallai-Edmonds decomposition X, Y, Z . In this section, we focus on the following problem: Given a set $\hat{S} \subseteq Y$, find a minimum cost

fractional additive stabilizer (M, y, c) among those which have $y_v = 1$ if $v \in \hat{S}$, and $y_v = \frac{1}{2}$ if $v \in Y \setminus \hat{S}$. Let $f(\hat{S}) = \sum_{e \in E} c_e$ denote the cost of such a solution.

We give an overview of the algorithm to compute $f(\hat{S})$. (For a formal description see Algorithm *MFASP*(\hat{S}).) Let (M, y, c) denote the triple of an optimal solution corresponding to $f(\hat{S})$. Let us examine the structure of the optimal solution (M, y, c) . Recall that we are restricting c and y to be half-integral.

Finding an optimum with knowledge of matching edges between Y and X . Let us focus on the matching edges in M that link Y to X and argue that it is sufficient to know these links to find an optimal solution c . We consider a component $K \in G[X]$ matched to some vertex in $v \in Y$ by M and distinguish two cases.

(i) K is non-trivial. By Proposition 12, we have $c(e) = 0$ for $e \in E[K]$ and $c(\delta(V(K))) = y_v - 1/2$.

(ii) Suppose $K = \{u\}$. If u is not incident to any vertex $w \in Y \setminus \hat{S}$, then we may assign $y_u = 0$ thereby incurring a cost of $c_e = 0$ and this is optimal. Otherwise, the feasible y assigns $y_u = 1/2$ and as before incurs an optimal cost of $y_v - 1/2$ over $\delta(V(K))$.

Next let us consider $K \in G[X]$ that is not matched to any vertex in Y .

(i) Suppose there are no edges $\{v, u\} \in \delta(V(K))$ that are incident to a vertex $u \in Y \setminus \hat{S}$. Then the optimal fractional additive stabilizer c restricted to the set of edges in $E(K) \cup \delta(V(K))$ should also be an optimal fractional additive stabilizer for K itself and vice-versa. Therefore, the stabilizer values on these edges can be computed using the exact algorithm for factor-critical graphs.

(ii) Suppose there are edges $\{v, u\} \in \delta(V(K))$ that are incident to a vertex $u \in Y \setminus \hat{S}$. In this case, the vertex cover values y should satisfy the covering constraints for the edges in $\delta(V(K))$. In particular, $y_v \geq 1/2$ for vertices $v \in V(K)$ which have neighbors in $Y \setminus \hat{S}$. As a consequence, the optimal stabilizer restricted to the set of edges in $E(K) \cup \delta(V(K))$ may not be the optimal stabilizer for K itself. However, the optimal fractional additive stabilizer restricted to the set of edges in $E(K) \cup \delta(V(K))$ should also be an optimal fractional additive stabilizer for a modified graph \tilde{K} obtained from K by adding an extra loop $\{v, v\}$ to each vertex $v \in V(K)$ linked to a vertex $u \in Y \setminus \hat{S}$. Conversely, we can modify an optimal fractional additive stabilizer c over the set of edges in $E(K) \cup \delta(V(K))$ to take the same values as an optimal fractional additive stabilizer for \tilde{K} without losing optimality. Further, we note that we can compute a minimum fractional additive stabilizer in \tilde{K} , by running the algorithm $A(v)$ for every node v that is not incident to a loop in \tilde{K} and output the best.

We observe that if every vertex $v \in V(K)$ has an edge adjacent to a vertex $u \in Y \setminus \hat{S}$, then this necessitates $y_v \geq 1/2$ for every vertex in K and therefore K must necessarily be matched to a vertex in Y .

Computing matching edges between Y and X . From the above discussion, it is clear that the cost of the solution $f(\hat{S})$ does not depend on the precise choice of the edges used to match the components of $G[X]$ to Y but only depends on which components of $G[X]$ are matched by M . Therefore, we can also identify the edges between Y and X in an optimal matching M as follows: Let us denote by $\kappa(K, \hat{S})$ the cost of the stabilizer over the edges in $E[K] \cup \delta(V[K])$ if K is not matched to Y (as observed before, we can compute $\kappa(K, \hat{S})$ by applying an

exact algorithm to the factor-critical graph \tilde{K} , for example the one presented in section 5). If K must necessarily be matched, then we set $\kappa(K, \hat{S})$ to infinity (or an arbitrarily large value U in the implementation). Let T denote the trivial components in $G[X]$ all of whose neighbors are in \hat{S} . Let us construct a weighted bipartite graph H from G as follows: Delete Z , delete the edges between vertices in Y , and contract each component K of $G[X]$ to a vertex v_K ; replace the multi-edges by a single edge to make it a simple graph and for a vertex $u \in Y$ that is adjacent to some node in K , we introduce weight $\kappa(K, \hat{S})$ on the edge $\{u, v_K\}$.

By the above discussion, a maximum weight matching N in H covering all vertices in Y gives the edges of an optimal matching M between Y and X . Therefore,

$$f(\hat{S}) = \frac{1}{2} \left(|\hat{S}| - |\{K \in T : N \text{ covers } v_K\}| \right) + \sum_{K \in G[X] : v_K \text{ is exposed by } N} \kappa(K, \hat{S}).$$

Hence, we find a maximum weight matching in H to compute $f(\hat{S})$.

Remark 26. Between matching a component in T or a non-trivial factor-critical component, M prefers the latter choice by Lemma 11. Thus, assigning $\kappa(K, \hat{S}) = 0$ for $K \in T$, implicitly assumes that components in T are only matched if there is no other choice.

Algorithm $MFASP(\hat{S})$.

1. For each factor-critical component K in $G[X]$ compute the cost $\kappa(K, \hat{S})$ needed to stabilize K and the edges linking K to vertices in $Y \setminus \hat{S}$ in case K would not be matched to Y . (We discussed above that this can be done in polynomial time.) Let (M^K, c^K, y^K) be an optimal stabilizer for $K \cup \delta(V(K))$ among those with $M^K \cap \delta(K) = \emptyset$.
2. Shrink the components K in $G[X]$ to pseudo-vertices v_K , assign the weight $\kappa(K, \hat{S})$ to all edges linking a Tutte vertex to pseudo-vertex v_K , and compute a bipartite matching \hat{M} of maximum weight covering Y (this is possible in polynomial time).

If no feasible solution exists, i.e. if there exists an unmatched component K with $\kappa(K, \hat{S}) = U$, STOP and RETURN INFEASIBLE.

3. Obtain a maximum matching in G by extending \hat{M} as follows:
 - for each component K not matched to Y add the matching edges in M^K to \hat{M} ;
 - for each component K having v_K matched to Y , pick a vertex in K that has the corresponding matching edge adjacent to it, say v , and add a maximum matching in K that exposes v to \hat{M} ;
 - for each component C in $G[Z]$ (we note that all these components are even), add an arbitrary perfect matching to \hat{M} ;
4. Obtain a fractional additive stabilizer as follows:
 - $\hat{c}_e = c_e^K$ for all components K in $G[X]$ that are not matched to Y ,

- $\hat{c}_e = \frac{1}{2}$ for each matching edge $e \in \hat{M}$ linked to a Tutte vertex $v \in \hat{S}$, except if $e = \{v, w\}$ for some vertex w that is a trivial component in $G[X]$ with $N_G(w) \subseteq \hat{S}$, and
 - $\hat{c}_e = 0$ else.
5. Obtain a fractional $(1 + \hat{c})$ -vertex cover \hat{y} that satisfies complementary slackness with \hat{M} as follows:
- $\hat{y}_v = \frac{1}{2}$ for all vertices in Z , all vertices in $Y \setminus \hat{S}$ and all vertices in components K in $G[X]$ that are matched to Y except if v is a trivial component in $G[X]$ with $N_G(v) \subseteq \hat{S}$,
 - $\hat{y}_v = 1$ for all $v \in \hat{S}$,
 - $\hat{y}_v = y_v^K$ for all vertices in components K in $G[X]$ that are not matched to Y , and
 - $\hat{y}_v = 0$ for all vertices v that are trivial components in $G[X]$ with $N_G(v) \subseteq \hat{S}$ and matched to Y .
6. Return $(\hat{M}, \hat{y}, \hat{c})$ and $f(\hat{S}) := \sum_{e \in E} \hat{c}_e$;

Remark 27. As mentioned earlier, not every possible choice of \hat{S} has a fractional additive stabilizer c which has $y_v = 1$ if $v \in \hat{S}$ and $y_v = 1/2$ if $v \in Y \setminus \hat{S}$ for a half-integral minimum fractional $(1 + c)$ -vertex cover y . For example, consider a graph where $Z = \emptyset$, $Y = \{v\}$ and $G[X]$ consists of two triangles whose nodes are all connected to v . Then y_v must be 1 and $\hat{S} = \emptyset$ is not feasible. The algorithm detects these cases in Step 2.

6.2 An Approximation Algorithm for Graphs with Many Nontrivial Components

We can also use the algorithm to compute $f(\hat{S})$ to obtain an approximation algorithm for graphs that have a large number of non-trivial factor-critical components in the Gallai-Edmonds decomposition.

Theorem 28. *For a graph G with Gallai-Edmonds decomposition $V(G) = X \cup Y \cup Z$, let \mathcal{C}^+ denote the set of nontrivial components in X . If $|\mathcal{C}^+| \geq (1 + 1/k)|Y|$ for $k > 0$, then there is a $(k/2 + 1)$ -approximation algorithm for MFASP.*

Proof. Let (M^*, y^*, c^*) be an optimal solution for MFASP with cost $|c^*|$ and X_1, \dots, X_r be the components of $G[X]$. We first note that the number of unmatched non-trivial components of X is at least $\lceil (1/k)|Y| \rceil$. We know by Lemma 11 that the optimal stabilizer pays at least 1 over the edges in each of these components. This yields

$$c^*(E) = \sum_{e \in E} c_e^* \geq \sum_{i=1}^r \sum_{e \in X_i} c_e^* \geq \frac{|Y|}{k}.$$

Let $\hat{S} = Y$, i.e., fix $y_t = 1$ for all vertices t in the Tutte set, and calculate an optimal solution $(\hat{M}, \hat{y}, \hat{c})$ corresponding to $f(\hat{S})$. We observe that the optimal solution corresponding to $f(\hat{S})$ can be computed efficiently using the algorithm

from Section 6.1. Recall that $y_v^* \geq 1/2$ for each vertex v in the Tutte set. Therefore,

$$\begin{aligned}\hat{c}(E) &= \hat{y}(V) - |\hat{M}| = \sum_{v \in V} \hat{y}_v - |M^*| = \sum_{v \in X} \hat{y}_v + \sum_{v \in Y} \hat{y}_v - |M^* \setminus E[Z]| \\ &\leq \sum_{v \in X} y_v^* + \frac{|Y|}{2} + \sum_{v \in Y} y_v^* - |M^* \setminus E[Z]| \\ &\leq \sum_{v \in X} y_v^* + \left(\frac{k}{2}\right) c^*(E) + \sum_{v \in Y} y_v^* - |M^* \setminus E[Z]| = \left(\frac{k}{2} + 1\right) c^*(E),\end{aligned}$$

which finishes the proof. In the first inequality above, we have used $\sum_{v \in X} \hat{y}_v \leq \sum_{v \in X} y_v^*$ since y^* restricted to X is also feasible for the auxiliary problem with y_v fixed to one on all Tutte vertices. \square

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